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The Theory of Sequential Relations*

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I. INTRODUCTION

Suppose that one has some system with inputs drawn from a set Σ^* which produces outputs in some set Δ^*. The system is assumed to have some state set S. Suppose that all possible experiments are run on the system in the sense that we can tabulate a relation R of all input-output pairs which the system can produce. (We will vary the starting state.) A problem of some importance in systems theory (Zadeh and Desoer, 1963) is to synthesize a system which has as its relation R where R has been "given" in advance.

In this paper, we study sequential relations which are the relations which arise in this manner when the system is a finite sequential machine. Our main results include synthesis procedures for constructing the sequential machine when the relation is "reasonably given." Also several characterizations of sequential relations are given as well as connections with other families of relations studied in automata theory.

This study was stimulated by a lecture given by Professor M. A. Aizermann in July 1964 at the University of California, Berkeley. Professor Aizermann described a method for the synthesis of sequential machines by a "question-answer" procedure. Since the technical details of the method were omitted, we derived a method which satisfied the general characteristics mentioned by Professor Aizermann. Recently, the original algorithm has become available (see Tal, 1964). It turns out that the two methods have very little in common. Our method is presented as an appendix to this paper.

Our method synthesizes a sequential function f by constructing the minimal sequential machine which computes f; this machine has a dis-

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tinguished initial state. Attempting to generalize our method to machines without initial states led naturally to sequential relations.

Professor L. A. Zadeh pointed out the relevance of the synthesis problem to the theory of systems. The present paper is the result of our investigations. Related results on sequential relations have been obtained independently by Gill (1966) and Deuel and Gill (1966).

II. BASIC PROPERTIES

We recall the definition of a sequential machine (Ginsburg, 1962).

DEFINITION. A sequential machine is a 5-tuple $M = \langle \Sigma, \Delta, S, f, g \rangle$

(i) Σ is a finite nonempty set (*inputs*).
(ii) Δ is a finite nonempty set (*outputs*).
(iii) S is a finite nonempty set (*states*).
(iv) f is a function from $S \times \Sigma$ into S (*direct transition functions*).
(v) g is a function from $S \times \Sigma$ into Δ (*output function*).

To describe the sequential operation of M, it is necessary to extend domain of the functions[1] f and g.

DEFINITION. Let $M = \langle \Sigma, \Delta, S, f, g \rangle$ be a sequential machine. For each $s \in S$, $\sigma \in \Sigma$, $x \in \Sigma^*$, define

$$f(s, \Lambda) = s \qquad g(s, \Lambda) = \Lambda$$

$$f(s, \sigma x) = f(f(s, \sigma), x) \qquad g(s, \sigma x) = g(s, \sigma)g(f(s, \sigma), x)$$

We now associate a relation with each sequential machine.

DEFINITION. Given a sequential machine $M = \langle \Sigma, \Delta, S, f, g \rangle$, the *relation computed by* M, denoted by $R(M)$, is defined as

$$R(M) = \bigcup_{s \in S} \bigcup_{x \in \Sigma^*} \{(x, g(s, x))\}$$

At this point, it is possible to define sequential relations which are the main topic of this paper.

DEFINITION. A relation $R \subseteq (\Sigma \times \Delta)^*$ is a *sequential relation over* $\Sigma \times \Delta$ (abbreviated $SR/(\Sigma \times \Delta)$) if there is a sequential machine M such that $R = R(M)$.

Convention. It is always understood that $\Sigma \neq \emptyset$ and $\Delta \neq \emptyset$.

An important variant of a sequential relation is a sequential function (Raney, 1958).

[1] Let X and Y be sets of words. The *product* of X and Y is $XY = \{xy \mid x \in X$ and $y \in Y\}$ where xy is the concatenation of x and y. Define $X^0 = \{\Lambda\}$ where Λ is the empty word and $X^{i+1} = X^iX$. $X^* = \bigcup_{i \geq 0} X^i$. Thus, if Σ is a finite nonempty set, Σ^* is the free monoid generated by Σ.

DEFINITION. Let $M = \langle \Sigma, \Delta, S, f, g \rangle$ be a sequential machine. Define the function $g_s(x) = g(s, x)$ for each $x \in \Sigma^*$. A function g from Σ^* into Δ^* is said to be a *sequential function* if there is a sequential machine $M = \langle \Sigma, \Delta, S, f, g \rangle$ and there is a state $s \in S$ such that

$$g(x) = g_s(x)$$

for each $x \in \Sigma^*$.

Remark. Every sequential relation which is a function is a sequential function. A sequential function need not be a sequential relation. (Cf. remark following Theorem 2.3.)

Our first result is an alternative characterization of sequential relations.

THEOREM 2.1. *A relation $R \subseteq (\Sigma \times \Delta)^*$ is a $SR/(\Sigma \times \Delta)$ if and only if there exist a finite number of sequential functions (over $\Sigma \times \Delta$), $g_1, \cdots, g_n$, such that*

(a) $R = \bigcup_{i=1}^{n} g_i$.

(b) *For each $\sigma \in \Sigma$ and each sequential function g_i, there exists a sequential function g_j such that for all $x \in \Sigma^*$, $g_i(\sigma x) = g_i(\sigma)g_j(x)$.*

Proof: The forward direction follows immediately from Definition 2.2. Conversely, assume that $R = \bigcup_{i=1}^{n} g_i$ where the g_i satisfy condition (b) above. We define $M = \langle \Sigma, \Delta, S, f, g \rangle$ where $S = \{g_i \mid i = 1, \cdots, n\}$. f is defined by the condition $f(g_i, \sigma) = g_j$ where $j = \min \{k \mid g_i(\sigma x) = g_i(\sigma)g_k(x)$ for all $x \in \Sigma^*\}$ while $g(g_i, \sigma) = g_i(\sigma)$.

Condition (b) guarantees that f is well defined. g is well defined since the g_i are functions, and M is therefore a sequential machine.

$$\begin{aligned} R(M) &= \bigcup_{s \in S} \bigcup_{x \in \Sigma^*} \{(x, g(s, x))\} \\ &= \bigcup_{i=1}^{n} \bigcup_{x \in \Sigma^*} \{(x, g_i(x))\} \\ &= \bigcup_{i=1}^{n} g_i \\ &= R \end{aligned}$$

Thus R is a $SR/(\Sigma \times \Delta)$ which concludes the proof.

In order to give alternative characterizations of sequential relations, some of the definitions from automata theory are required.

DEFINITION. A *finite automaton* is a 5-tuple $A = \langle \Sigma, S, M, a, F \rangle$ where

(i) Σ is a finite nonempty set (*inputs*).

(ii) S is a finite nonempty set (*states*).

(iii) $a \in S$ is the *initial state.*

(iv) $F \subseteq S$ is the set of *final states*.

(v) M is a function from $S \times \Sigma$ into S (*direct transition function*).

The domain of M is extended to $S \times \Sigma^*$ by the following definition: $M(s, \Lambda) = s$ and $M(s, \sigma x) = M(M(s, \sigma), x)$ for all $s \in S, \sigma \in \Sigma, x \in \Sigma^*$.

DEFINITION. Let $A = \langle \Sigma, S, M, a, F \rangle$ be a finite automaton and let $x \in \Sigma^*$. A is said to *accept* or *recognize* x if $M(a, x) \in F$. The set of all sequences accepted by A is denoted by $T(A)$ and is called the *behavior* of A. A set $R \subseteq \Sigma^*$ is called Σ-*regular* if there is a finite automaton $A = \langle \Sigma, S, M, a, F \rangle$ such that $R = T(A)$.

The following definitions will also be needed in the characterization.

DEFINITION. A set $R \subseteq \Sigma^*$ is *prefix closed* if for all $x, y \in \Sigma^*$, $xy \in R$ implies $x \in R$. x is called a *prefix* of xy. Similarly $R \subseteq \Sigma^*$ is *suffix closed* if $xy \in R$ implies $y \in R$ for all $x, y \in \Sigma^*$. y is called a *suffix* of xy.

Notation. For $x \in \Sigma^*$, $\lg(x)$ denotes the *length* of x.

DEFINITION. A relation $R \subseteq \Sigma^* \times \Delta^*$ is said to be *length preserving* if for each $(x, y) \in R$, $\lg(x) = \lg(y)$.

DEFINITION. A length preserving relation R is *prefix closed* if $(x_1x_2, y_1y_2) \in R$ and $\lg(x_1) = \lg(y_1)$ implies $(x_1, y_1) \in R$.

The following definition is originally due to Elgot, (1961).

DEFINITION. A relation $R \subseteq (\Sigma \times \Delta)^*$ is said to be *strongly extendable over* Σ if for each $(x, y) \in R$ and each $\sigma \in \Sigma$, there exists $\delta \in \Delta$ such that $(x\sigma, y\delta) \in R$. A relation $R \subseteq (\Sigma \times \Delta)^*$ is said to be *functionally extendable* if for each $(x, y) \in R$ and $\sigma \in \Sigma$, there is a unique $\delta \in \Delta$ such that $(x\sigma, y\delta) \in R$.

Using the above definitions, Elgot has given a natural characterization of sequential functions. Elgot's theorem 7.1 is quoted below.[2]

THEOREM 2.2. (Elgot, 1961). *Let* $\Sigma, \Delta \neq \emptyset$. $R \subseteq \Sigma^* \times \Delta^*$ *is a sequential function over* Σ *if and only if*

(a) R *is* $(\Sigma \times \Delta)$-*regular.*

(b) R *if a function.*

(c) $dom(R) = \Sigma^*$ (*cf. footnote* 3).

(d) R *is prefix-closed.*

Remark. Note that the condition (a) implies that R is length preserving.

The following proposition summarizes a number of basic properties of sequential relations. The proof is omitted.

[2] In Elgot's paper, the "Moore-model" of a sequential machine is used while we use a "Mealy model." Since it is well known that those models are equivalent, Elgot's theorem carries over to the present discussion.

[3] dom (R) denotes the *domain* of the relation R, i.e., dom $(R) = \{x \mid (x, y) \in R\}$.

PROPOSITION 2.1. *If* $R \subseteq \Sigma^* \times \Delta^*$ *is a* $SR/(\Sigma \times \Delta)$, *then*

(a) R *is* $(\Sigma \times \Delta)$*-regular.*

(b) R *is length preserving.*

(c) R *is prefix and suffix closed.*

(d) R *is strongly extendable over* Σ.

(e) *If* $\Delta \subseteq \Gamma$, *then* R *is* $SR/(\Sigma \times \Gamma)$.

(f) R *is infinite.*

We now give another characterization of sequential relations. The proof will depend on a lemma to be proven in the next section, namely, that sequential relations are closed under union (Lemma 3.3).

THEOREM 2.3. *Let* $R \subseteq (\Sigma \times \Delta)^*$. R *is a* $SR/(\Sigma \times \Delta)$ *if and only if*

(a) *there are sequential functions,* $g_1, \cdots, g_n$, *over* $\Sigma \times \Delta$ *such that* $R = \bigcup_{i=1}^{n} g_i$, *and*

(b) R *is suffix closed.*

Proof: If R is a $SR/(\Sigma \times \Delta)$, then (a) and (b) follow from Theorem 2.1 and Proposition 2.1(c). Conversely, assume that $R = \bigcup_{i=1}^{n} g_i \subseteq (\Sigma \times \Delta)^*$ is suffix closed and that each g_i is a sequential function over $\Sigma \times \Delta$. Since g_i is a sequential function, there is a machine $M_i = \langle \Sigma, \Delta, S_i, f_i, h_i \rangle$ with the following properties: There is a state $s_i \in S_i$ such that

(i) $g_i(x) = h_i(s_i, x)$ for each $x \in \Sigma^*$.

(ii) For each $t \in S_i$, there is some $x \in \Sigma^*$ such that $f_i(s_i, x) = t$.

We shall complete the proof by showing $R = \bigcup_{i=1}^{n} R(M_i)$. By construction, $g_i \subseteq R(M_i)$ so that

$$R = \bigcup_{i=1}^{n} g_i \subseteq \bigcup_{i=1}^{n} R(M_i).$$

Conversely, suppose $(x, y) \in \bigcup_{i=1}^{n} R(M_i)$. Then $(x, y) \in R(M_k)$ for some $k (1 \leqq k \leqq n)$. There is some state $t \in S_k$ such that $h_k(t, x) = y$. By (ii) above, there exists $u \in \Sigma^*$ such that $f_k(s_k, u) = t$.

Let $v \in \Delta^*$ be defined as $v = h_k(s_k, u)$. Thus

$$\begin{aligned} g_k(ux) &= h_k(s_k, ux) \\ &= h_k(s_k, u) h_k(f_k(s_k, u), x) \\ &= h_k(s_k, u) h_k(t, x) = vy. \end{aligned}$$

Therefore, $(ux, vy) \in g_k \subseteq \bigcup_{i=1}^{n} g_i = R$. Since R is suffix closed, $(x, y) \in R$. This completes the proof that $\bigcup_{i=1}^{n} R(M_i) = R$. Clearly, the $R(M_i)$ are $SR/(\Sigma \times \Delta)$, so by Lemma 3.3, R is a $SR/(\Sigma \times \Delta)$.

Remark. Note that when $n = 1$ in Theorem 2.3, the following result is obtained. A sequential function g is a sequential relation if and only if g is suffix closed.

III. CLOSURE PROPERTIES

In this section we obtain a number of results concerning the operations under which sequential relations are closed. See (Elgot-Mezei, 1965) for some related results.

Notation. $R(\Sigma, \Delta)$ denotes the set of all sequential relations over $\Sigma \times \Delta$.

Our results are summarized in Table I.

We now proceed to verify the first column of Table I.

LEMMA 3.1. *If*[4] $|\Delta| = 1$, *then* $R \subseteq (\Sigma \times \Delta)^*$ *is a* $SR/(\Sigma \times \Delta)$ *if and only if* $R = (\Sigma \times \Delta)^*$.

Proof: Let $\Delta = \{\delta\}$. It is clear that $(\Sigma \times \{\delta\})^*$ is computed by a one state sequential machine. Conversely, let R be a $SR/(\Sigma \times \{\delta\})$. By Theorem 2.1, $R = \bigcup_{i=1}^{n} g_i$ where each g_i is a sequential function. For each i, $g_i = (\Sigma \times \{\delta\})^*$ since g_i is length preserving and dom $g_i = \Sigma^*$ by Theorem 2.2 (a) and (c).

To justify column 1 of Table I, note that Lemma 3.1 immediately implies that $R(\Sigma, \{\delta\})$ is closed under union, intersection, product, and transpose. Moreover, $R(\Sigma, \{\delta\})$ cannot be closed under complementation or symmetric difference.

Remark. $R(\Sigma, \{\delta\})$ is closed under converse if and only if $\Sigma = \{\delta\}$.

Next, we consider composition of sequential relations

LEMMA 3.2. *Let* $\Delta \subseteq \Sigma$ *and* R, S *be* $SR/(\Sigma \times \Delta)$. *Then* $R \circ S$ *is a* $SR/(\Sigma \times \Delta)$.

Proof: Let $R = R(M_1)$ where $M_1 = \langle \Sigma, \Delta, S_1, f_1, g_1 \rangle$ and $S = R(M_2)$ where $M_2 = \langle \Sigma, \Delta, S_2, f_2, g_2 \rangle$. Assume, without loss of generality, that $S_1 \cap S_2 = \emptyset$. Construct $M = \langle \Sigma, \Delta, S_1 \cup S_2, f, g \rangle$ where $f((s_1, s_2), \sigma) = (f_1(s_1, \sigma), f_2(s_2, \sigma))$ and $g((s_1, s_2), \sigma) = g_2(s_2, g_1(s_1, \sigma))$ for each $(s_1, s_2) \in S_1 \times S_2$ and each $\sigma \in \Sigma$. Thus

$$\begin{aligned} R(M) &= \bigcup_{(s_1, s_2) \in S_1 \times S_2} \bigcup_{x \in \Sigma^*} \{(x, g((s_1, s_2), x))\} \\ &= \bigcup_{s_1 \in S_1} \bigcup_{s_2 \in S_2} \bigcup_{x \in \Sigma^*} \{(x, g_2(s_2, g_1(s_1, x)))\} \\ &= \bigcup_{s_2 \in S_2} \bigcup_{x \in \Sigma^*} \{(x, g_2(s_2, y)) \mid (x, y) \in R\} \\ &= \bigcup_{x \in \Sigma^*} \{(x, z) \mid (x, y) \in R \text{ and } (y, z) \in S \text{ for some } y \in \Delta^*\} \\ &= R \circ S. \end{aligned}$$

[4] $|\Delta|$ denotes the *cardinality* of Δ.

TABLE I
CLOSURE PROPERTIES OF $SR/(\Sigma \times \Delta)$

Is $R(\Sigma, \Delta)$ closed under	$\|\Delta\| = 1$(4)	$\|\Delta\| > 1$
Complementation (relative to $(\Sigma \times \Delta)^*$)	no	no
Union	yes	yes
Intersection	yes	no
Symmetric difference	no	no
Composition[a] (when $\Delta \subseteq \Sigma$)	yes	yes
Converse[b] (when $\Delta = \Sigma$)	yes	no
Product[c]	yes	no
*	yes	no
Transpose[d]	yes	no

[a] Let $\Delta \subseteq \Sigma$. The *composition* of two binary relations $R_1 \subseteq \Sigma^* \times \Delta^*$ is the relation $R_1 \circ R_2 \subseteq \Sigma^* \times \Delta^*$ defined by $R_1 \circ R_2 = \{(x, z) \mid (x, y) \in R_1 \text{ and } (y, z) \in R_2 \text{ for some } y \in \Delta^* \subseteq \Sigma^*\}$.

[b] Let $R \subseteq \Sigma^* \times \Delta^*$. The *converse* of R, denoted by R^c, is $R^c = \{(y, x) \mid (x, y) \in R\}$.

[c] Let $R_1, R_2 \subseteq \Sigma^* \times \Delta^*$. The *product* of R_1 and R_2 is $R_1R_2 = \{(x_1x_2, y_1y_2)\} (x_1, y_1) \in R_1$ and $(x_2, y_2) \in R_2\}$.

[d] Let $x \in \Sigma^*$. The *transpose* of x, written x^T, is defined by $\Lambda^T = \Lambda$, and $(y\sigma)^T = \sigma y^T$ for each $y \in \Sigma^*$ and $\sigma \in \Sigma$. Let $R \subseteq \Sigma^* \times \Delta^*$. The *transpose* of R, written R^T, is $R^T = \{(x^T, y^T) \mid (x, y) \in R\}$.

Therefore $R \circ S$ is a $SR/(\Sigma \times \Delta)$ which concludes the proof.

We now verify that sequential relations are closed under union.

LEMMA 3.3. *If $R, S \subseteq (\Sigma \times \Delta)^*$ are $SR/(\Sigma \times \Delta)$, then $R \cup S$ is a $SR/(\Sigma \times \Delta)$.*

Proof: Let $R = R(M_1)$ where $M_1 = \langle \Sigma, \Delta, S_1, f_1, g_1 \rangle$ and $S = R(M_2)$ where $M_2 = \langle \Sigma, \Delta, S_2, f_2, g_2 \rangle$. Assume without loss of generality that $S_1 \cap S_2 = \emptyset$. Construct $M = \langle \Sigma, \Delta, S_1 \cup S_2, f, g \rangle$ where $f = f_1 \cup f_2$ and $g = g_1 \cup g_2$. Since $S_1 \cap S_2 = \emptyset$, f, and g are functions. Moreover[5] $f \restriction (S_i \times \Sigma) = f_i$ and $g \restriction (S_i \times \Sigma) = g_i$ for $i = 1, 2$. It is easy to verify that

$$\begin{aligned} R(M) &= \bigcup_{s \in S_1 \cup S_2} \bigcup_{x \in \Sigma^*} \{(x, g(s, x))\} \\ &= \Big(\bigcup_{s_1 \in S_1} \bigcup_{x \in \Sigma^*} \{(x, g(s, x))\}\Big) \cup \Big(\bigcup_{s \in S_2} \bigcup_{x \in \Sigma^*} \{(x, g(s, x))\}\Big) \\ &= \Big(\bigcup_{s \in S_1} \bigcup_{x \in \Sigma^*} \{(x, g_1(s, x))\}\Big) \cup \Big(\bigcup_{s \in S_2} \bigcup_{x \in \Sigma^*} \{(x, g_2(s, x))\}\Big) \\ &= R \cup S. \end{aligned}$$

[5] If f is any function from X into Y and $A \subseteq X$, then the *restriction* of f to A, written $f \restriction A$, is defined as $f \restriction A = f \cap (A \times Y)$.

We have completed our verification of the affirmative entries of Table I, column 2. A lemma is now established which will be used in verifying the negative entries in column 2 of Table I.

LEMMA 3.4. *If* $|\Delta| > 1$, *then* $(\Sigma \times \Delta)^*$ *is not a* $SR/(\Sigma \times \Delta)$.

Proof: Suppose that $(\Sigma \times \Delta)^*$ is a $SR/(\Sigma \times \Delta)$. There exist sequential functions $g_1, \cdots, g_n$ such that $(\Sigma \times \Delta)^* = \bigcup_{i=1}^n g_i$. Let m be the least nonnegative integer such that $|\Delta|^m > n$. (m must exist since $|\Delta| \geqq 2$.) Let $x \in \Sigma^*$ with $\lg(x) = m$ be chosen. Define the set $A_x = \{(x, y) | (x, y) \in (\Sigma \times \Delta)^*\}$. Clearly $|A_x| = |\Delta|^m > n$, but $A_x \subseteq \bigcup_{i=1}^n g_i$. Since $|A_x| > n$, there must exist an integer k such that $1 \leqq k \leqq n$ and $(x, y) \in g_k$ and $(x, z) \in g_k$ for some $y \neq z$. But g_k is a function. This contradiction establishes that $(\Sigma \times \Delta)^*$ is not a $SR/(\Sigma \times \Delta)$.

We now begin to apply the previous lemma.

PROPOSITION 3.1. *Let* $R \subseteq (\Sigma \times \Delta)^*$. *Either* R *or* $\bar{R}$ *is not a* $SR/(\Sigma \times \Delta)$.

Proof: Assume that both R and $\bar{R}$ are $SR/(\Sigma \times \Delta)$. If $|\Delta| = 1$, then $R = (\Sigma \times \Delta)^*$ and $\bar{R} = \emptyset$. Therefore $\bar{R}$ is not a $SR/(\Sigma \times \Delta)$ which establishes the case $|\Delta| = 1$. If $|\Delta| > 1$, then by Lemma 3.3, $R \cup \bar{R} = (\Sigma \times \Delta)^*$ is a $SR/(\Sigma \times \Delta)$ which contradicts Lemma 3.4. So in any case the proposition holds.

We now establish that in general sequential relations are not closed under product.

PROPOSITION 3.2. *If* $|\Delta| > 1$ *then* $R(\Sigma, \Delta)$ *is not closed under product.*

Proof: Let $\{\delta, \gamma\} \subseteq \Delta$ since $|\Delta| \geqq 2$. Let $R = (\Sigma \times \{\delta\})^*$ and $S = (\Sigma \times \{\delta\})^*$. By Lemma 3.1 and Proposition 2.1(e), R and S are $SR/(\Sigma \times \Delta)$. Assume that RS is a $SR/(\Sigma \times \Delta)$. There must exist sequential functions, $g_1, \cdots, g_n$, such that $RS = \bigcup_{i=1}^n g_i$. Let $x \in \Sigma^*$ such $\lg(x) = n$ and define $A = \{(x, \delta^i\gamma^{n-i}) | 0 \leqq i \leqq n\}$. Since $|A| = n + 1$ and $A \subseteq RS$, there exist integers i, j, k, with the properties

(i) $0 \leqq i, j \leqq n$.

(ii) $1 \leqq k \leqq n$.

(iii) $i \neq j$.

(iv) $(x, \delta^i\gamma^{n-i}) \in f_k$ and $(x, \delta^j\gamma^{n-j}) \in f_k$.

Condition (iv) contradicts the fact that f_k is a function and completes the proof.

Remark. Using R and S as in the proof of Proposition 3.2, note that $R + R = R \cap S = \emptyset$. Thus $R(\Sigma, \Delta)$ is not closed under symmetric difference or under intersection.

PROPOSITION 3.3. *If* $|\Delta| > 1$, *then* $R(\Delta, \Delta)$ *is not closed under converse*

Proof: Let $\{\delta, \gamma\} \subseteq \Delta$ since $|\Delta| \geqq 2$. By Lemma 3.1 and Proposition 2.1(e), $R = (\Delta \times \{\delta\})^*$ is a $SR/(\Delta \times \Delta)$. Assume that $R^c = (\{\delta\} \times \Delta)^*$ is a $SR/(\Delta \times \Delta)$. By the remark preceding Lemma 3.2, $\Delta = \{\delta\}$. This contradicts that $|\Delta| > 1$.

Next, we consider closure under $*$.

PROPOSITION 3.4. *If* $|\Delta| > 1$, *then* $R(\Sigma, \Delta)$ *is not closed under* $*$.

Proof: Let $\{\delta, \gamma\} \subseteq \Delta$ where $\delta \neq \gamma$. $R = (\Sigma \times \{\delta\})^*$ and $S = (\Sigma \times \{\gamma\})^*$ are $SR/(\Sigma \times \Delta)$. By Lemma 3.3, $R \cup S$ is a $SR/(\Sigma \times \Delta)$. Clearly $(\Sigma \times \{\delta, \gamma\}) \subseteq R \cup S \subseteq (\Sigma \times \{\delta, \gamma\})^*$.
Thus

$$(\Sigma \times \{\delta, \gamma\})^* \subseteq (R \cup S)^* \subseteq (\Sigma \times \{\delta, \gamma\})^*.$$

Therefore $(R \cup S)^* = (\Sigma \times \{\delta, \gamma\})^*$. By Lemma 3.4, $(\Sigma \times \{\delta, \gamma\})^*$ is not a $SR/(\Sigma \times \Delta)$. This establishes the proposition.

Lastly, we establish nonclosure under transpose.

PROPOSITION 3.5. *If* $|\Delta| > 1$, *then* $R(\Sigma, \Delta)$ *is not closed under transpose.*

Proof: Let $\{\delta, \gamma\} \subseteq \Delta$ since $|\Delta| \geqq 2$. The set

$$R = (\Sigma \times \{\delta\})(\Sigma \times \{\gamma\})^* \cup (\Sigma \times \{\gamma\})^*$$

is a $SR/(\Sigma \times \Delta)$. (This is shown by constructing a two-state sequential machine M such that $R = R(M)$. The construction is omitted.) Assume that

$$R^T = (\Sigma \times \{\gamma\})^*(\Sigma \times \{\delta\}) \cup (\Sigma \times \{\gamma\})^*$$

is a $SR/(\Sigma \times \Delta)$. Let $\sigma \in \Sigma$, then $(\sigma, \delta) \in R^T$. However, for each $\delta' \in \Delta$, $(\sigma\sigma, \delta\delta') \notin R^T$. Thus R^T is not strongly extendable so by Proposition 2.1(d), R^T is not a $SR/(\Sigma \times \Delta)$.

We have now completed the detailed verification of Table I.

There are many equivalent characterizations of the regular subsets of Σ^*. (Cf. (Rabin-Scott, 1959).) It is known that for the regular subsets of other monoids (in particular, $\Sigma^* \times \Delta^*$), that these various characterizations are no longer equivalent. Elgot and Mezei (1965) have presented a number of new characterizations. We now give a characterization of $(\Sigma \times \Delta)$-regular sets ($= FAD$ sets in the terminology of Elgot and Mezei (1965)) using sequential relations.

DEFINITION. Let D be any family of subsets of $(\Sigma \times \Delta)^*$. The *regular closure*[6] $K(D)$ of D is the least family of subsets of $(\Sigma \times \Delta)^*$ such that

[6] $K(D)$ is different from the Kleenian closure of Elgot and Mezei.

$D \subseteq K(D)$ and $K(D)$ is closed under union, intersection, product, $*$, and complementation.[7]

The following alternative definition of the regular closure will be used in the induction proof of Theorem 3.2.

DEFINITION. Let D be any family of subsets of $(\Sigma \times \Delta)^*$. Define $D_0 = D$ and for all $i \geqq 0$. $D_{i+1} = \{R \mid R = R_1^* \text{ for some } R_1 \in D_i\}$

$$\cup \{R \mid R = R_1 \cup R_2 \quad \text{for some} \quad R_1, R_2 \in D_i\}$$

$$\cup \{R \mid R = R_1 R_2 \quad \text{for some} \quad R_1, R_2 \in D_i\}$$

$$\cup \{R \mid R = R_1 \cap R_2 \quad \text{for some} \quad R_1, R_2 \in D_i\}$$

$$\cup \{R \mid R = \bar{R}_1 = (\Sigma \times \Delta)^* - R_1 \quad \text{for some} \quad R_1 \in D_i\}.$$

It is easy to verify that $D_i \subseteq D_{i+1}$ for all $i \geqq 0$ and that $K(D) = \bigcup_{i=0}^{\infty} D_i$.

Notation. Let $\mathfrak{I}(\Sigma \times \Delta) = \{A \subseteq (\Sigma \times \Delta)^* \mid A \text{ is } (\Sigma \times \Delta)\text{-regular}\}$.

The following basic result is due to Kleene. (Cf. Rabin-Scott (1959) and Harrison (1965) for proofs.)

THEOREM 3.1. (Kleene)

(a) $K(\mathfrak{I}(\Sigma \times \Delta)) = \mathfrak{I}(\Sigma \times \Delta)$.

(b) $\mathfrak{I}(\Sigma \times \Delta) = K(\{\{(\sigma, \delta)\} \mid (\sigma, \delta) \in \Sigma \times \Delta\})$.

The following result is a set-theoretic characterization of $\Sigma \times \Delta$ regular sets in terms of sequential relations.

THEOREM 3.2.

(a) *If* $|\Delta| = 1$, *then* $K(R(\Sigma, \Delta)) = \{(\Sigma \times \Delta)^*, \emptyset\}$.

(b) *If* $|\Delta| > 1$, *then* $K(R(\Sigma, \Delta)) = \mathfrak{I}(\Sigma \times \Delta)$.

Proof: If $|\Delta| = 1$, then (a) follows immediately from Lemma 3.1.

If $|\Delta| > 1$, let $\{\delta, \gamma\} \subseteq \Delta$ and define $A = \{\{(\sigma, \delta)\} \mid (\sigma, \delta) \in \Sigma \times \Delta\}$. We shall show that $K(A) \subseteq K(R(\Sigma, \Delta))$ by induction. For each $(\sigma, \delta) \in \Sigma \times \Delta$ and each $\delta' \in \Delta - \{\delta\}$, define the relation R (which depends on σ, δ, and δ') as follows $R = [(\{(\sigma, \delta)\} \cup ((\Sigma - \{\sigma\}) \times \{\delta'\})) \cup \{(\Lambda, \Lambda)\}][\Sigma \times \{\delta'\}]^*$. R is a $SR/(\Sigma \times \Delta)$.[8] Define $S = (\Sigma \cup \{\delta\})^*$; S is a $SR/(\Sigma \times \Delta)$ by Lemma 3.1 and Proposition 2.1(e).

Observe that $\emptyset \in K(R(\Sigma, \Delta))$ since it is the union of none of the elements of $R(\Sigma, \Delta)$. Since $\emptyset^* = \{(\Lambda, \Lambda)\}$, we have that $\{(\Lambda, \Lambda)\} \in K(R(\Sigma, \Delta))$.

[7] Complementation is relative to $(\Sigma \times \Delta)^*$, i.e., $\bar{A} = (\Sigma \times \Delta)^* - A$.

[8] $R = R(M)$ where $M = \langle \Sigma, \Delta, \{s_1, s_2\}, f, g \rangle$ where $f(s_1, \sigma') = f(s_2, \sigma') = s_2$ for all $\sigma' \in \Sigma$. Also $g(s_1, \sigma) = \delta$ while $g(s_1, \sigma') = \delta'$ for all $\sigma' \in \Sigma - \{\sigma\}$ and $g(s_2, \sigma') = \delta'$ for each $\sigma' \in \Sigma$.

$$R \cap S - \{(\Lambda, \Lambda)\} = \{(\sigma, \delta)\}.$$

Therefore $A \subseteq K(R(\Sigma, \Delta))$ and

$$K(A) \subseteq K(K(R(\Sigma, \Delta))) = K(R(\Sigma, \Delta)). \tag{1}$$

By Theorem 3.1, $K(A) = \mathfrak{I}(\Sigma \times \Delta)$ and by Proposition 2.1(a) $R(\Sigma, \Delta) \subseteq \mathfrak{I}(\Sigma \times \Delta)$. Thus

$$K(R(\Sigma, \Delta)) \subseteq K(\mathfrak{I}(\Sigma \times \Delta)) = \mathfrak{I}(\Sigma \times \Delta). \tag{2}$$

By (1), (2), and Theorem 3.1(b)

$$\mathfrak{I}(\Sigma \times \Delta) = K(A) \subseteq K(R(\Sigma, \Delta)) \subseteq \mathfrak{I}(\Sigma \times \Delta).$$

Therefore $K(R(\Sigma, \Delta)) = \mathfrak{I}(\Sigma \times \Delta)$.

At this point, we establish a result (also due independently to Deuel and Gill (1966)) which asserts that it is decidable whether two sequential machines have the same relation.

THEOREM 3.3. *Let $R(M_1)$ and $R(M_2)$ be $SR/(\Sigma \times \Delta)$. It is decidable whether $R(M_1) = R(M_2)$.*

Proof: Let $M_i = \langle \Sigma, \Delta, S_i, f_i, g_i \rangle$ where $i = 1, 2$. Let d_1 and d_2 be new symbols. Define $A_i = \langle \Sigma \times \Delta, S_i \cup \{d_i\}, N_i, S_i, S_i \rangle$ where for each $(\sigma, \delta) \in \Sigma \times \Delta$, $s \in S_i$,

$$N_i(s, (\sigma, \delta)) = \begin{cases} t & \text{if } f_i(s, \sigma) = t \text{ and } g_i(s, \sigma) = \delta \\ d_i & \text{otherwise.} \end{cases}$$

Also $N_i(d_i, (\sigma, \delta)) = d_i$. Clearly $T(A_i) = R(M_i)$. Since it is decidable whether $T(A_1) = T(A_2)$ (Rabin-Scott, 1959), it is decidable whether $R(M_1) = R(M_2)$ because our construction is effective.

IV. FINITE AUTOMATA AND SEQUENTIAL RELATIONS

In Section V, a number of decidability and undecidability results will be given. In order to prove these results, certain facts about the finite automata which accept sequential relations are needed. (Recall Proposition 2.1(a).)

DEFINITION. Let $A = \langle \Sigma, S, M, a, F \rangle$ be a finite automaton. A is said to be *connected* if for each $s \in S$, there is an $x \in \Sigma^*$ such that $M(a, x) = s$.

Well known algorithms exist for converting any finite automaton A into an equivalent connected automaton. (Cf. (Harrison, 1965, Chap. 9).)

We now prove a long sequence of lemmas which ultimately lead to our main synthesis theorem.

LEMMA 4.1. *Let* $A = \langle \Sigma \times \Delta, S, M, a, F\rangle$ *be a connected finite automaton. If*

(a) $T(A)$ *is a* $SR/(\Sigma \times \Delta)$.

(b) $s \in F$.

(c) *There exists* $(\sigma x, \delta y) \in (\Sigma \times \Delta)^*$ *such that* $M(s, (\sigma x, \delta y)) = s$, *then* $M(s, (\sigma, \gamma)) \notin F$ *for each* $\gamma \in \Delta - \{\delta\}$.

Proof: Since $A = \langle \Sigma \times \Delta, S, M, a, F\rangle$ is connected, for each $s \in F$, there is $(u, v) \in (\Sigma \times \Delta)^*$ such that $M(a, (u, v)) = s$. By (c), there is $(\sigma x, \delta y) \in (\Sigma \times \Delta)(\Sigma \times \Delta)^*$ such that $M(s, (\sigma x, \delta y)) = s$. Thus

$$M(a, (u(\sigma x)^n, v(\delta y)^n)) = s \quad \text{for each} \quad n \geqq 0.$$

Assume that $M(s, (\sigma, \gamma)) \in F$ for some $\gamma \in \Delta - \{\delta\}$. Then $M(a, (u(\sigma x)^n\sigma, v(\delta y)^n\gamma)) \in F$ for each $n \geqq 0$. Thus $(u(\sigma x)^{n+1}, v(\delta y)^{n+1}) \in T(A)$ and $(u(\sigma x)^n\sigma, v(\delta y)^n\gamma) \in T(A)$. By (a), $T(A)$ is a $SR/(\Sigma \times \Delta)$ and hence prefix closed (Proposition 2.1(c)). Therefore

$$(u(\sigma x)^n\sigma, v(\delta y)^n\delta) \in T(A)$$

and

$$(u(\sigma x)^n\sigma, v(\delta y)^n\gamma) \in T(A).$$

Since $T(A)$ is a $SR/(\Sigma \times \Delta)$, there are sequential functions $g_1, \cdots, g_n$ such that

$$T(A) = \bigcup_{i=1}^{n} g_i.$$

Define, for $k \geqq 0$,

$$i_k = \min \{i \mid (u(\sigma x)^k\sigma, v(\delta y)^k\gamma) \in g_i\}$$

and consider the sequence $(i_0, i_1, \cdots)$. We shall prove that if $p \neq q$, then $i_p \neq i_q$. This will contradict the finiteness of n.

Let $p \neq q$ and assume without loss of generality, that $p < q$. Then

$$(u(\sigma x)^q\sigma, v(\delta y)^q\gamma) \in g_{i_q}.$$

Since g_{i_q} is prefix closed (Theorem 2.2(d)), for each $0 \leqq j < q$, $(u(\sigma x)^j\sigma, v(\delta y)^j\delta) \in g_{i_q}$. Because g_{i_q} is a function, $(u(\sigma x)^i\sigma, v(\delta y)^i\gamma) \notin g_{i_q}$ for every $0 \leqq i < q$. However $p < q$ and $(u(\sigma x)^p\sigma, v(\delta y)^p\gamma) \in g_{i_p}$ so $g_{i_p} \neq g_{i_q}$. Thus all the i_k are distinct which contradicts the finiteness of n and completes the proof.

Remark. If the phrase "sequential relation" is replaced by "sequential function" everywhere in Lemma 4.1, the resulting statement is still valid.

The next lemma is a statement about those finite automata which accept sets which are prefix closed.

LEMMA 4.2. *Let* $A = \langle \Sigma, S, M, a, F \rangle$ *be a connected finite automaton.* $T(A)$ *is prefix closed if and only if for each* $(s, \sigma) \in (S - F) \times \Sigma$, $M(s, \sigma) \notin F$.

Proof: Suppose $T(A)$ is prefix closed and assume there is some $(s, \sigma) \in (S - F) \times \Sigma$ such that $M(s, \sigma) \in F$. Since A is connected, there exists $y \in \Sigma^*$ with $M(a, y) = s$. Thus $M(a, y\sigma) \in F$ so $y\sigma \in T(A)$. Because $T(A)$ is prefix closed, $y \in T(A)$. This contradicts that $s = M(a, y) \notin F$.

Suppose that $M(s, \sigma) \notin F$ for all $(s, \sigma) \in (S - F) \times \Sigma$. Assume that there exist words $x, y \in \Sigma^*$ such that $xy \in T(A)$ and $x \in T(A)$. Let $y = \sigma_1 \cdots \sigma_n$ where $\sigma_i \in \Sigma$, and $n \geqq 1$. Let i be the least number such that $x\sigma_1 \cdots \sigma_i \notin T(A)$ and $x\sigma_1 \cdots \sigma_{i+1} \in T(A)$. Then let $t = M(a, x\sigma_1 \cdots \sigma_i)$. Clearly $t \in S - F$ while $M(t, \sigma_{i+1}) \in F$. This is a contradiction, and thus the lemma is established.

COROLLARY 1. *Let* $A = \langle \Sigma, S, M, a, F \rangle$ *be a connected finite automaton.* $T(A)$ *is prefix-closed if and only if for each* $s \in S - F$ *and* $x \in \Sigma^*$, $M(s, x) \notin F$.

Proof: The argument is a trivial induction on lg (x) and is omitted.

COROLLARY 2. *Let* $A = \langle \Sigma, S, M, a, F \rangle$ *be a connected minimal*[9] *finite automaton. If for each* $(s, x) \in (S - F) \times \Sigma^*$, $M(s, x) \notin F$, *then* $T(A)$ *is prefix-closed if and only if* $| S - F | = 1$.

Proof: Suppose $T(A)$ is prefix closed and assume $s, t \in S - F$ with $s \neq t$. By hypothesis, $M(s, x) \notin F$ and $M(t, x) \notin F$ for each $x \in \Sigma^*$. Hence s and t are equivalent which contradicts that A is minimal.

Suppose $S - F = \{s\}$ and assume $xy \in T(A)$ with $x \notin T(A)$. Therefore $M(a, x) \in S - F$ so $M(a, x) = s$. But $M(s, y) = M(a, xy) \in F$ is a contradiction.

The following lemma which characterizes sequential functions will prove to be very useful later.

[9] Let $A = \langle \Sigma, S, M, a, F \rangle$ be a finite automaton. Two states $s, t \in S$ are said to be *equivalent* if for each $x \in \Sigma^*$, $M(s, x) \in F$ if and only if $M(t, x) \in F$. A is said to be *minimal* if no pair of distinct states are equivalent. It is well known (Harrison, 1965, Chap. 11) that there are algorithms for constructing the minimal automaton, A^M, of A. Moreover $T(A^M) = T(A)$.

LEMMA 4.3. *Let* $A = \langle \Sigma \times \Delta, S, M, a, F \rangle$ *be a connected finite automaton.* $T(A)$ *is a sequential function if and only if*

(a) $a \in F$.

(b) $T(A)$ *is prefix closed.*

(c) *For each* $(s, \sigma) \in F \times \Sigma$, *there is a unique* $\delta \in \Delta$ *such that* $M(s, (\sigma, \delta)) \in F$.

Proof: If $T(A)$ is a sequential function, $(\Lambda, \Lambda) \in T(A)$ so that (a) is proven. By Theorem 2.2(d), $T(A)$ is prefix closed so that (b) is established. Suppose that (c) is false. There exists $(s, \sigma) \in F \times \Sigma$ such that (d) for each $\delta \in \Delta, M(s, (\sigma, \delta)) \notin F$ or (e) there are $\delta, \gamma \in \Delta$ such that $\gamma \neq \delta$, $M(s, (\sigma, \delta)) \in F$, and $M(s, (\sigma, \gamma)) \in F$. Suppose (d) holds. Since A is connected, there exists $(x, y) \in (\Sigma \times \Delta)^*$ such that $M(a, (x, y)) = s$. By (d), $M(a, (x\sigma, y\delta)) \notin F$, for all δ. Thus $(x\sigma, y\delta)) \notin T(A)$. Next we must show that $(x\sigma, y'\delta') \notin T(A)$ with $y \neq y'$. Suppose it were. Since $T(A)$ is prefix closed, $(x, y') \in T(A)$ which contradicts that $T(A)$ is a function. Therefore $x\sigma \notin \text{dom}\,(T(A))$ which contradicts Theorem 2.2(c). Thus (d) cannot hold.

Suppose that (e) holds. Recalling that there is some (x, y) such that $M(a, (x, y)) = s$, we have

$$(x\sigma, y\delta) \in T(A) \quad \text{and} \quad (x\sigma, y\gamma) \in T(A).$$

Since $\delta \neq \gamma$, $T(A)$ is not a function. This contradiction finishes the forward direction of the proof.

Suppose that A satisfies (a), (b), and (c). Then $T(A)$ is $(\Sigma \times \Delta)$-regular and prefix closed. By Theorem 2.2, it suffices to show that $\text{dom}\,(T(A)) = \Sigma^*$ and that $T(A)$ is a function. It is easy to show the following proposition by induction on the length of x (using (a), (b), and (c)).

For each $x \in \Sigma^*$, there is a unique $y \in \Delta^*$ such that

$$\lg(x) = \lg(y) \quad \text{and} \quad (x, y) \in T(A). \tag{$*$}$$

From $(*)$, both the functionality of $T(A)$ and the fact $\text{dom}\,(T(A)) = \Sigma^*$ follow. The proof is complete.

We shall next present an involved construction which is used to convert a given finite $(\Sigma \times \Delta)$ automaton into a sequential machine with input set Σ and output set Δ. While the details of the construction are involved, the basic idea is quite simple. We shall illustrate this idea as follows:

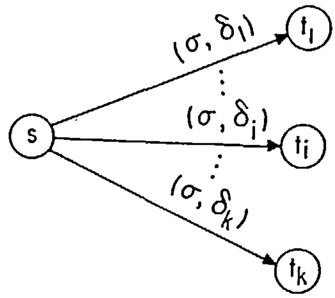


FIG. 1. Part of the automaton A in which $s, t_i \in F$ and the states $t_1, \cdots, t_{i-1}, t_{i+1}, \cdots, t_k$ (t_j may equal t_k when $j \neq k$) may or may not be in F.

Let $(s, \sigma) \in F \times \Sigma$ be fixed and let $\Delta = \{\delta_i \mid 1 \leqq i \leqq k\}$. Suppose we fix $\delta_i \in \Delta$ and let $S - F = \{t\}$. Assume that part of the automaton A is as shown in Fig. 1.

The construction which we shall introduce, called the (s, σ)-reduction, will produce a sequence of automata. The ith automaton in the sequence will have been altered as shown in Fig. 2.

A study of Fig. 1 indicates the basic technique. We take an automaton in which there are at least two branches whose labels start with σ leading from s to other states. These branches are broken to leave only one such branch while all other branches go to a "unique dead state." This technique will be iterated in all possible ways. The following definition formalizes the construction.

DEFINITION. Let $A = \langle \Sigma \times \Delta, S, M, a, F \rangle$ be a connected minimal finite automaton in which $S - F = \{t\}$. Let $F' \supseteq F$ be a finite set and let $(s, \sigma) \in F' \times \Sigma$. If (a) for each $\delta \in \Delta$ and each $(x, y) \in (\Sigma \times \Delta)^*$, $M(s, (\sigma x, \delta y)) \neq s$ and (b) $B = \{\delta \in \Delta \mid M(s, (\sigma, \delta)) \in F\}$ has at[10] least two elements, then define the sequence of automata (A_δ') where $\delta \in B$ and

$$A_\delta' = \langle \Sigma \times \Delta, S, M_\delta, a, F \rangle$$

where

$$\begin{aligned} M_\delta = M \upharpoonright [((S - \{s\}) \times \Sigma \times \Delta) \cup (\{s\} \times (\Sigma - \{\sigma\}) \times \Delta)] \\ \cup \{s\} \times \{\sigma\} \times (\Delta - \{\delta\}) \times \{t\} \cup \{(s, (\sigma, \delta), M(s, (\sigma, \delta)))\}. \end{aligned}$$

[10] B is defined with respect to a fixed $(s, \sigma) \in F' \times \Sigma$. We write B instead of $B_{s,\sigma}$.

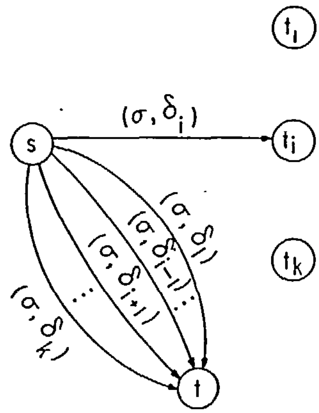


FIG. 2. The part of the ith automaton in the (s, σ)-reduction of A which corresponds to Fig. 1.

Define the sequence (A_δ) where $\delta \in B$ as the sequence obtained from A_δ' by taking each A_δ as the connected minimal machine associated with A_δ'. The (s, σ)-*reduction* of A is defined to be sequence (A_δ), $\delta \in B$. If A violates (a) or (b) or if $s \notin F$, then A is said to be (s, σ)-*irreducible*. (In this case (A_δ) is (A).)

Remark. In each automaton A_δ', only one change has been made. Part of the machine of the form given in Fig. 1 has been converted to a part given in Fig. 2.

We wish to iterate the construction.

DEFINITION. Let A be as in the previous definition and let $T \subseteq F \times \Sigma$. If $|T| = n$, index T by $\{i \mid 1 \leqq i \leqq n\}$ in any arbitrary (but henceforth fixed) way. A *T-reduction*[11] *of* A, written S^n, is defined recursively by

$$S^0 = (A).$$

If (s, σ) is the ith element of $T(i \geqq 1)$ and if

$$S^{i-1} = (A_1, \cdots, A_k)$$

then

$$S^i = (A_{11}, \cdots, A_{1n(A_1,i)}, \cdots, A_{k1}, \cdots, A_{kn(A_k,i)})$$

where for each $1 \leqq p \leqq k$, $(A_{p1}, \cdots, A_{pn(A_p,i)})$ is the (s, σ)-reduction

[11] T-reductions are not necessarily unique.

of A_p. If $T = F \times \Sigma$, and for each $(s, \sigma) \in T$, A is (s, σ)-irreducible, then A is *T-irreducible.*

The properties of a reduction of A are now derived.

LEMMA 4.4. *Let $A = \langle \Sigma \times \Delta, S, M, a, F \rangle$ satisfy the following conditions*

(i) *A is connected and minimal.*

(ii) $S - F = \{t\}$.

(iii) *$T(A)$ is prefix closed.*

(iv) *For each $s \in F$, $\sigma \in \Sigma$; $\delta, \gamma \in \Delta$, and $(x, y) \in (\Sigma \times \Delta)^*$, $M(s, (\sigma x, \delta y)) = s$ implies $\gamma = \delta$ or $M(s, (\sigma, \gamma)) \notin F$.*

Let $(s, \sigma) \in F \times \Sigma$. Construct the (s, σ)-reduction of A, written $(A_1, \cdots, A_m)$ where $A_i = \langle \Sigma \times \Delta, S_i, N_i, a, F_i \rangle$. Then the reduction has the following properties.

(v) *Each A_i is connected and minimal.*

(vi) $S_i - F_i = \{t\}$.

(vii) *For each $\sigma \in \Sigma$; $\delta, \gamma \in \Delta$, and $(x,y) \in (\Sigma \times \Delta)^*$ $N_i(s, (\sigma x, \delta y)) = s$ implies $\gamma = \delta$ or $N_i(s, (\sigma, \gamma)) \notin F_i$.*

(viii) *$T(A_i)$ is prefix closed.*

(ix) $T(A) = \bigcup_{i=1}^{m} T(A_i)$.

Proof: Since each A_i is in the (s, σ)-reduction of A, the A_i are connected and minimal. This proves (v).

By (ii) and the reduction procedure, we have $S_i - F_i = \{t\}$, for each i. This establishes (vi).

By Corollary 1 to Lemma 4.2, for each $(x, y) \in (\Sigma \times \Delta)^*$

$$M(t, (x, y)) \notin F.$$

By the reduction algorithm,

$$N_i(t, (x, y)) \notin F_i \quad \text{for each } i.$$

By Corollary 1 of Lemma 4.2, $T(A_i)$ is also prefix closed. Therefore (viii) is established.

The following statement is useful in establishing both (vii) and (ix). This statement can be easily proven by induction on the length of $(x, y) \in (\Sigma \times \Delta)^*$.

(x) *For each $(x,y) \in (\Sigma \times \Delta)^*$, $s \in F$, and for each i, $N_i(a, (x, y)) = s$ implies $M(a, (x, y)) = s$.*

To establish (vii), assume that there are $\delta, \gamma \in \Delta$, $\delta \neq \gamma$, there is $(x, y) \in (\Sigma \times \Delta)^*$, and there is $s \in F_i$ such that $N_i(s, (\sigma x, \delta y)) = s$ and $N_i(s, (\sigma, \gamma)) \in F_i$. Using (x) twice establishes that

$$M(s, (\sigma x, \delta y)) = s$$

and that $M(s, (\sigma, \gamma)) \in F$. By (iv), either $\gamma = \delta$ (which is impossible) or $M(s, (\sigma, \gamma)) \in F$ (which is a contradiction). Thus (vii) has been proven.

From (x), it immediately follows that $T(A_i) \subseteq T(A)$ for each i, hence

(xi) $\bigcup_{i=1}^{m} T(A_i) \subseteq T(A)$.

To prove the converse of (xi), suppose state s violates condition (a) of the definition of reduction. Then A is its own (s, σ)-reduction and the converse of (xi) is established. If state s satisfies part (a) of the definition, let $x = x_1 \cdots x_n \in T(A)$ where each $x_i \in (\Sigma \times \Delta)$. If

$$s \notin \{M(a, x_1 \cdots x_i) \mid 0 \leqq i \leqq n\},$$

then $M(a, x) = N_i(a, x)$ for all $i = 1, \cdots, n$. This follows from the definition of the (s, σ)-reduction. If $s \in \{M(a, x_1 \cdots x_i) \mid 0 \leqq i \leqq n\}$, then define $s_i = M(a, x_1 \cdots x_i)$ for $i = 1, \cdots, n$. Let $s_0 = a$, then $M(s_i, x_{i+1}) = s_{i+1}$ for $0 \leqq i < n$. Let j be the least integer such that $s_j = s$ *and* there exists $\delta \in \Delta$ such that $x_{j+1} = (\sigma, \delta)$. (If j does not exist, then $x \in T(A_i)$ for each $i = 1, \cdots, m$ because the "part" of A used in accepting x is unaffected by the (s, σ)-reduction.)

If j exists, then the (s, σ)-reduction of A contains a unique machine A_k with the property $N_k(s, (\sigma, \delta)) \in F_k$. By the definition of the (s, σ)-reduction, the following two properties, (xii) and (xiii) hold.

(xii) *For each* i $(i = 1, \cdots, m)$, *each* $u \in F$, *and each* $(\sigma', \delta) \in (\Sigma - \{\sigma\}) \times \Delta$, $N_i(u, (\sigma', \delta)) = M(u, (\sigma', \delta))$.

(xiii) *For each* i, $(i = 1, \cdots, m)$, *each* $u \in F - \{s\}$, *and each* $(\sigma', \delta) \in \Sigma \times \Delta$, $N_i(u, (\sigma', \delta)) = M(u, (\sigma', \delta))$.

Using the existence of j, (xii), and (xiii), we have $M(s_l, x_{l+1}) = s_{l+1} = N_i(s_l, x_{l+1})$ for each $0 \leqq l < j$ and all $1 \leqq i \leqq m$. Then $M(a, x_1 \cdots x_{j+1}) = N_k(a, x_1 \cdots x_{j+1})$.

By condition (a) of the definition of reduction, $M(a, x_1 \cdots x_p) \neq s$ for each p such that $j < p \leqq n$. Therefore

$$M(a, x_1 \cdots x_n) = N_k(a, x_1 \cdots x_n).$$

Since $x = x_1 \cdots x_n \in T(A)$, $x \in T(A_k)$, and the converse of (xi) is established, this completes the proof of Lemma 4.4.

In our applications of the reduction, the process is iterated. The following lemma completes our description of the automata which arise in this construction. The main property to be established is that when an automaton A which recognizes a sequential relation is reduced to irreducible components A_i, then each $T(A_i)$ is a sequential function and $T(A) = \bigcup_i T(A_i)$ (cf. Theorem 2.3).

LEMMA 4.5. *Let* $A = \langle \Sigma \times \Delta, S, M, a, F\rangle$ *be a connected minimal finite automaton. Let* $F \times \Sigma$ *be indexed by* $\{i \mid 1 \leqq i \leqq n\}$ *where* $n = |F \times \Sigma|$. *Assume* $T(A) \neq (\Sigma \times \Delta)^*$ *is a* $SR/(\Sigma \times \Delta)$. *Define the following sequence of sequences of automata*

$$S^0 = (A).$$

For each $i \geqq 1$, *if* $S^{i-1} = (A_1, \cdots, A_m)$, *then*

$$S^i = (A_{11}, \cdots, A_{1n(A_1,i)}, \cdots, A_{m1}, \cdots, A_{mn(A_m,i)})$$

where if (s, σ) *is the* ith *member of* $F \times \Sigma$, *then the* (s, σ)-*reduction of* A_j *is* $(A_{j1}, \cdots, A_{jn(A_j,i)})$ *for each* $1 \leqq j \leqq m$.

If $S^h = (A_1, \cdots, A_k)$, *and if for all* $1 \leqq i \leqq k$, $A_i = \langle \Sigma \times \Delta, S_i, N_i, a, F_i\rangle$ *then*

(i) A_i *is connected and minimal.*

(ii) $S - F = \{t\}$ *and* $S_i - F_i = \{t\}$ *for each* i.

(iii) *For each* $s \in F_i$, $\sigma \in \Sigma$; $\gamma, \delta \in \Delta$, *and each* $(x, y) \in (\Sigma \times \Delta)^*$, $N_i(s, (\sigma x, \delta y)) = s$ *implies* $\gamma = \delta$ *or* $N_i(s, (\sigma, \gamma)) \in F_i$.

(iv) $T(A_i)$ *is prefix closed for each* i.

(v) $T(A) = \bigcup_{i=1}^k T(A_i)$.

(vi) *If* $S^n = (A_1, \cdots, A_p)$, *i.e.*, S^n *is a* $F \times \Sigma$ *reduction of* A, *then for each* $1 \leqq i \leqq p$, $T(A_i)$ *is a sequential function.*

Proof: Since $T(A)$ is a $SR/(\Sigma \times \Delta)$, then $T(A)$ is prefix closed. Since A is connected and minimal and since $T(A) \neq \emptyset$, there is a unique state t such that $S - F = \{t\}$. When $h = 1$, Lemmas 4.1 and 4.4 imply (i)–(v). Assume that (i) through (v) hold for $1 \leqq h \leqq j < n$. Consider $h = j + 1$. If $S^j = (A_1, \cdots, A_k)$ and $S^h = S^{j+1} = (A_{11}, \cdots, A_{1n(A_1,h)}, \cdots, A_{k1}, \cdots, A_{kn(A_k,h)})$ by Lemma 4.4 we see that

(vii) *For each* $1 \leqq i \leqq k$, $T(A_i) = \bigcup_{p=1}^{n(A_i,h)} T(A_{i,p})$

and for each $1 \leqq i \leqq k$, and $1 \leqq p \leqq n(A_i, h)$ we see that A_{ip} satisfies conditions (v)–(viii) of Lemma 4.4 and thus conditions (i) through (iv) of the present lemma. By the induction hypothesis,

$$T(A) = \bigcup_{i=1}^k T(A_i).$$

By (vii) $T(A) = \bigcup_{i=1}^k \bigcup_{p=1}^{n(A_i,h)} T(A_{i,p})$ which establishes (v).

For each A_i appearing in S^n, $T(A_i)$ is prefix closed and $a \in F_i$. We will show that

(viii) *For each* $(s, \sigma) \in (F_i \times \Sigma)$, *there is a unique* $\delta \in \Delta$, $N_i(s, (\sigma, \delta)) \in F_i$.

To establish this we perform a case analysis which involves the structure of A. Suppose $(s, \sigma) \in (F \times \Sigma)$ is fixed. Then, three alternatives are possible.

(ix) *There is a unique* $\delta \in \Delta$ *such that* $M(s, (\sigma, \delta)) \in F$, *or*

(x) *For all* $\delta \in \Delta$, $M(s, (\sigma, \delta)) \notin F$, *or*

(xi) *There are* $\delta, \gamma \in \Delta, \gamma \neq \delta, M(s, (\sigma, \delta)) \in F$ *and* $M(s, (\sigma, \gamma)) \in F$.

If s satisfies (ix), then by the definition of the $F \times \Sigma$ reduction of A, the (s, σ) reduction is trivial (i.e., $S^{i-1} = S^i$ if (s, σ) is the ith element of $F \times \Sigma$). So in each A_i if $s \in F_i$, then $M(s, (\sigma, \delta)) = N_i(s, (\sigma, \delta))$ for each δ. So by (ix), there is a unique δ such that $N_i(s, (\sigma, \delta)) \in F_i$ and thus (viii) holds.

Suppose that (x) is true. Since A is connected there is $(x, y) \in (\Sigma \times \Delta)^*$ such that $M(a, (x, y)) = s$. Further, $M(a, (x\sigma, y\delta)) \notin F$ for all δ. Since $s \in F$, $(x, y) \in T(A)$. But for all $\delta \in \Delta$, $(x\sigma, y\delta) \notin T(A)$. This contradicts Proposition 2.1(d) since $T(A)$ is a $SR/(\Sigma \times \Delta)$. So (x) can never hold.

Suppose that (xi) is true. For each $\delta \in \Delta$, and $(x, y) \in (\Sigma \times \Delta)^*$, $M(s, (\sigma x, \delta y)) \neq s$. (Because if not, by Lemma 4.1, $M(s, (\sigma, \gamma)) \notin F$ which contradicts (xi).) Therefore the (s, σ)-reduction of A is nontrivial. That is, if (s, σ) is the ith member of $(F \times \Sigma)$, then $S^{i-1} \neq S^i$. Now if $S^i = (A_1^i, \cdots, A_l^i)$ where $A_k^i = \langle \Sigma \times \Delta, S_k^i, N_k^i, a, F_k^i \rangle$, then for each A_k^i of S^i, there is a unique $\delta \in \Delta$ such that

$$N_k^i(s, (\sigma, \delta)) \in F_k^i.$$

But since no (σ, δ)-branch leaving state s is altered in succeeding reductions, we see that for each $1 \leqq j \leqq h$, if $s \in F_j$, there is a unique $\delta \in \Delta$ such that $N_j(s, (\sigma, \delta)) \in F_j$. But (s, σ) was an arbitrary member of $F \times \Sigma$ so (viii) is established.

Thus $T(A)$ is a sequential function by Lemma 4.3. The proof is now complete.

Remark. The entire reduction process is easily seen to be effective.

Remark. The $F \times \Sigma$ reduction depends on the ordering of $F \times \Sigma$. This reduction may not be unique. In general, the decomposition of $T(A)$ into $T(A_i)$ (where each $T(A_i)$ is a sequential function) is the "finest possible decomposition." (Gill, 1966) has shown that there are minimal nonisomorphic machines with the same relation. In general, the machines produced by the reduction may be "relationally equivalent" to smaller machines. (Cf. (Gill, 1966).) Not every minimal sequential machine with relation $T(A)$ is produced in a $F \times \Sigma$ reduction from A.

We now obtain the major result of this section.

THEOREM 4.1. *Let* $A = \langle \Sigma \times \Delta, S, M, a, F \rangle$ *be a connected minimal finite automaton.*

(i) *If* $|\Delta| = 1$, *then* $T(A)$ *is a sequential relation if and only if* $S = F$. *Let* $|\Delta| > 1$ *and* $S - F = \{t\}$. *Let a* $(F \times \Sigma)$*-reduction of* A *be* $(A_1, \cdots, A_n)$.

(ii) $T(A)$ *is a* $SR/(\Sigma \times \Delta)$ *if and only if:*

(iii) *For each* $1 \leqq i \leqq n$, $T(A_i)$ *is a sequential function.*

(iv) $T(A)$ *is suffix closed.*

Proof: (i) follows from Lemma 3.1. Let $|\Delta| > 1$. Suppose $T(A)$ is a $SR/(\Sigma \times \Delta)$. By Theorem 2.3 and Lemma 4.5 (vi), (iii) and (iv) follow. If (iii) and (iv) hold, then by Theorem 2.3 and the fact (Lemma 4.5) that $T(A) = \bigcup_{i=1}^{n} T(A_i)$ we conclude that $T(A)$ is a sequential function.

V. RECOGNITION OF SEQUENTIAL RELATIONS

We now consider the problem of testing a given relation $R \subseteq (\Sigma \times \Delta)^*$ to decide whether or not R is a $SR/(\Sigma \times \Delta)$. If R is a $SR/(\Sigma \times \Delta)$, an algorithm for constructing a sequential machine M such that $R = R(M)$ is desired. These problems are the "recognition problem" and "synthesis problem" respectively.

Since R must be infinite, it is not clear how R is "given." In order to make the recognition problem meaningful, it is necessary to assume that R is given by some finite set of rules for its generation. (For example, by a sequential machine M with $R = R(M)$, although this would make the recognition and synthesis problems trivial.)

First we consider that R is given as the behavior of some finite automaton (over $\Sigma \times \Delta$).

THEOREM 5.1. *If* $R = T(A)$ *where* $A = \langle \Sigma \times \Delta, S, M, s, F \rangle$ *is a given connected minimal finite automaton, then there is an effective procedure for deciding whether or not* R *is a* $SR/(\Sigma \times \Delta)$.

Proof: If $|\Delta| = 1$, R is a $SR/(\Sigma \times \Delta)$ if and only if $S = F$. (Cf. Theorem 4.1.) Since both S and F are finite sets, this test is effective.

If $|\Delta| > 1$, then R is a $SR/(\Sigma \times \Delta)$ if and only if

(i) $|S - F| = 1$.

(ii) R is suffix closed.

(iii) For each A_i in a $(\Sigma \times F)$-reduction of A, $T(A_i)$ is a sequential function.
(Again Theorem 4.1.) It is clear that (i) can be effectively decided since S and F are finite.

To show that it is possible to decide whether or not R is suffix closed, let suf (R) (respectively pref (R)) denote the set of all suffixes (prefixes) of R. It is well known that pref (R) is regular if R is regular (Elgot, 1961, Lemma 6.3). But suf $(R) = (\text{pref } (R^T))^T$, hence suf (R) is regular. Finite automata recognizing these sets can be effectively constructed. Thus R is suffix closed if and only if suf $(R) \subseteq R$. Thus (ii) is decidable since containment of regular sets is a decidable relation.

Using Lemma 4.5, each $T(A_i)$ in a $F \times \Sigma$ reduction of A is a sequential function. By Lemma 4.3, $T(A_i)$ is a sequential function if and only if

(iv) $a \in F$.

(v) *For each* $(s, \sigma) \in F \times \Sigma$, *there is a unique* $\delta \in \Delta$ *such that* $M(s, (\sigma, \delta)) \in F$.

(vi) $T(A)$ *is prefix-closed.*

By the finiteness of F and $F \times \Sigma \times \Delta$ (iv) and (v) are all effective. Further, Lemma 4.2 and the finiteness of $(S - F) \times \Sigma \times \Delta$ shows that condition (vi) can be effectively tested. So condition (iii) is effective and the proof is complete.

We now start the proof of our principal results which give a synthesis method for constructing sequential machines having a prescribed sequential relation. Two preliminary lemmas are required.

LEMMA 5.1. *Let* $A = \langle \Sigma \times \Delta, S, N, a, F\rangle$ *be a connected minimal finite automaton such that* $T(A)$ *is a sequential function. Then one may effectively construct a sequential machine* $M = \langle \Sigma, \Delta, F, f, g\rangle$ *such that* $R(M) = \mathit{suf}\ (T(A))$.[12]

Proof: By Lemma 4.3, for each $(s, \sigma) \in F \times \Sigma$, there is a unique $\delta \in \Delta$ with $N(s, (\sigma, \delta)) \in F$. Define $M = \langle \Sigma, \Delta, F, f, g\rangle$ where f and g are defined by the following condition. For all $(s, \sigma, \delta) \in F \times \Sigma \times \Delta$,

(i) $f(s, \sigma) = s'$ *and* $g(s, \sigma) = \delta$ *if and only if* $N(s, (\sigma, \delta)) = s' \in F$.

By Lemma 4.3, f and g are total functions on $F \times \Sigma$ so that M is a sequential machine. Further by the finiteness of Σ, Δ, and F, f and g are finite sets so the construction is effective. Condition (i) may be extended to condition (ii) below by an obvious inductive argument which is omitted.

[12] Recall that for $X \subseteq (\Sigma \times \Delta)^*$, suf (X) is the set of all suffixes of elements of X. X is suffix closed if and only if suf $(X) = X$.

For each $(s, (x, y)) \in F \times (\Sigma \times \Delta)^*$,

(ii) *$f(s, x) = s'$ and $g(s, x) = y$ if and only if $N(s, (x, y)) = s' \in F$.*

To show that suf $(T(A)) = R(M)$, suppose $(x, y) \in$ suf $(T(A))$. There exist $(u, v) \in (\Sigma \times \Delta)^*$ with $(ux, vy) \in T(A)$ or equivalently, $N(a, (ux, vy)) \in F$. Since $a \in F$ (Lemma 4.3 again), $g(a, ux) = vy$. Then $(ux, vy) \in R(M)$. Since $R(M)$ is suffix closed (Proposition 2.1(c)), $(x, y) \in R(M)$.

Conversely, assume $(x, y) \in R(M)$. There exists $s \in F$ such that $g(s, x) = y$. Let $s' = f(s, x)$. Then by (ii), $N(s, (x, y)) = s' \in F$. Since A is connected, there is $(u, v) \in (\Sigma \times \Delta)^*$ such that

$$N(a, (u, v)) = s.$$

Thus $N(ux, vy)) \in F$ so that $(ux, vy) \in T(A)$. Therefore $(x, y) \in$ suf $(T(A))$ and the proof is complete.

The next lemma explains how the reduction technique is used to obtain a sequential machine which has a given relation.

LEMMA 5.2. *Let $A = \langle \Sigma \times \Delta, S, N, a, F \rangle$ be a connected minimal finite automaton. If $T(A)$ is a $SR/(\Sigma \times \Delta)$, then there is a sequential machine M such that $T(A) = R(M)$ and M may be effectively constructed from A.*

Proof: Let $(A_1, \cdots, A_n)$ be a $F \times \Sigma$ reduction of A. By Lemma 5.1, for each A_i, we can effectively construct sequential machines $M_i = \langle \Sigma, \Delta, S_i', f_i, g_i \rangle$ such that $R(M_i) =$ suf $(T(A_i))$. Assume, without loss of generality, that the S_i' are pairwise disjoint. Define $M = \langle \Sigma, \Delta, T, f, g \rangle$ where $T = \bigcup_{i=1}^n S_i'$, $f = \bigcup_{i=1}^n f_i$, and $g = \bigcup_{i=1}^n g_i$. The finiteness of the S_i', f_i, and g_i insure that the construction is effective. By Lemma 3.3 (more exactly, by the construction in the proof) and by induction, it is clear that M is a sequential machine and that

$$R(M) = \bigcup_{i=1}^{n} R(M_i).$$

However

$$R(M) = \bigcup_{i=1}^{n} R(M_i) = \bigcup_{i=1}^{n} \text{suf } (T(A_i)).$$

Since, for any family of sets X, $\bigcup_{x \in X}$ suf $(x) =$ suf $(\bigcup_{x \in X} x)$

$$R(M) = \text{suf } \left(\bigcup_{i=1}^{n} T(A_i)\right) = \text{suf } (T(A))$$

by Lemma 4.5(v). $T(A)$ is suffix closed (by Proposition 2.1(c)) so suf $(T(A)) = T(A)$ and

$$R(M) = T(A).$$

The entire construction is effective since the reduction is effective.

The previous lemmas yield our main synthesis result.

THEOREM 5.2. *If a relation R is given as $R = T(A)$ where $A = \langle \Sigma \times \Delta, S, N, a, F \rangle$ is a finite automaton, then there is an algorithm for deciding if R is a $SR/(\Sigma \times \Delta)$ and if R is a $SR/(\Sigma \times \Delta)$, there is an effective procedure for constructing a sequential machine M such that $R = T(A) = R(M)$.*

Proof: Theorem 5.1 and Lemma 5.2.

COROLLARY 1. *Given $R = T(A)$ and $S = T(B)$ where A and B are $(\Sigma \times \Delta)$-finite automaton, then it is decidable if $\bar{R}$, R^T, R^*, RS, R^c, $R \times S$, $R \times S$, $R + S$ are $SR/(\Sigma \times \Delta)$. Moreover if any of these sets are $SR/(\Sigma \times \Delta)$, then an associated sequential machine may be effectively constructed.*

Proof: The result follows from the closure of regular sets under these operations and from Theorems 5.1 and 5.2.

Infinite sets are often denoted by (finite) expressions in some formal language. The example of regular expressions and regular events is well known in automata theory.[13] We shall assume that a given relation R is a *transduction*[14] in the sense of Elgot and Mezei and we shall represent it by a *transduction expression* which we now define.

For simplicity in the following definition, we take $\Sigma = \{0, 1\} = \Delta$. The extension to arbitrary Σ and Δ will be obvious.

DEFINITION. *A transduction expression* over $\Sigma^* \times \Delta^*$ where $\Sigma = \Delta = \{0, 1\}$ is defined as follows.

(i) $\emptyset$ is a transduction expression.

(ii) $(0, \Lambda)$ and $(1, \Lambda)$ are transduction expressions.

(iii) $(\Lambda, 0)$ and $(\Lambda, 1)$ are transduction expressions.

(iv) If α, β are transduction expressions, then $\alpha \cup \beta$ is a transduction expression.

(v) If α, β are transduction expressions, then $\alpha\beta$ is a transduction expression.

[13] The treatment in (Harrison, 1965) adopts this point of view and is assumed to be familiar to the reader.

[14] Let $\mathfrak{F}$ denote the set of all finite relations over $\Sigma^* \times \Delta^*$. A transduction is any relation contained in the least set containing $\mathfrak{F}$ and closed under finite union, product, and $*$.

(vi) If α is a transduction expression, then α^* is a transduction expression.

(vii) There are no tranduction expressions except those given by (i)–(vi).

The interpretation of transduction expressions is defined in the conventional way as follows.

DEFINITION. For each transduction α, we define the set denoted by α, written $\| \alpha \|$ as follows.

(i) $\| \emptyset \| = \emptyset$

(ii) $\| (0, \Lambda) \| = \{(0, \Lambda)\}$ and $\| (1, \Lambda) \| = \{(1, \Lambda)\}$

(ii) $\| (\Lambda, 0) \| = \{(\Lambda, 0)\}$ and $\| (\Lambda, 1) \| = \{(\Lambda, 1)\}$

(iv) $\| \alpha \cup \beta \| = \| \alpha \| \cup \| \beta \|$

(v) $\| \alpha\beta \| = \| \alpha \| \| \beta \|$

(vi) $\| \alpha^* \| = \| \alpha \|^*$.

The following proposition is stated without proof since its justification is implicitly given in Elgot and Mezei (1965).

THEOREM 5.3. *$R \subseteq \Sigma^* \times \Delta^*$ is a transduction if and only if there exists a transduction expression α such that $R = \| \alpha \|$.*

Elgot and Mezei (1965) have shown that the transductions are exactly the sets accepted by nondeterministic automata.[15]

The following lemma is needed in deciding whether a transduction expression is length-preserving.

LEMMA 5.3. *Let α, β, and γ be transduction expressions where $\| \alpha \| \neq \emptyset$ and $\| \gamma \| \neq \emptyset$. $\| \alpha\beta^*\gamma \|$ is length-preserving if and only if*

(i) $\| \alpha\gamma \|$ is length-preserving and $\| \beta^* \| = \{(\Lambda, \Lambda)\}$, or

(ii) $\| \alpha\gamma \|$ is length-preserving and $\| \beta^* \| \neq \{(\Lambda, \Lambda)\}$ and $\| \beta \|$ is length-preserving.

Proof: Sufficiency is obvious. Suppose that $\| \alpha\beta^*\gamma \|$ is length-preserving. If $\| \beta^* \| = \{(\Lambda, \Lambda)\}$ (i.e., $\| \beta \| = \emptyset$ or $\{(\Lambda, \Lambda)\}$), then $\| \alpha\gamma \|$ must be length-preserving.

Suppose $\| \beta^* \| \neq \{(\Lambda, \Lambda)\}$, then there exists $(w, x) \in \| \beta^* \|$ where $(w, x) \neq (\Lambda, \Lambda)$. Since $\| \alpha \| \neq \emptyset$ and $\| \gamma \| \neq \emptyset$, let $(u, v) \in \| \alpha \|$ and $(y, z) \in \| \beta \|$. Because $(uwy, vxz) \in \| \alpha\beta^*\gamma \|$ and $\| \alpha\beta^*\gamma \|$ is length-preserving, we have that $\lg(u) + \lg(w) + \lg(y) = \lg(v) + \lg(x) + \lg(z)$. Suppose that $\lg(w) \neq \lg(x)$. Then

$$\lg(u) + 2\lg(w) + \lg(y) \neq \lg(v) + 2\lg(x) + \lg(z).$$

[15] A *nondeterministic automaton* is a 6-tuple $A = \langle \Sigma, \Delta, S, \nu, a, F \rangle$ where Σ and Δ are finite nonempty sets of *inputs* and *outputs* respectively. S is a finite nonempty set of *states* while $a \in S$ and $F \subseteq S$. ν is a finite subset of $S \times \Sigma^* \times \Delta^* \times S$.

On the other hand $(ww, xx) \in \| \beta^* \|$ so that this contradicts that $\| \alpha\beta^*\gamma \|$ is length-preserving. Therefore $\lg(w) = \lg(x)$ for any $(w, x) \in \| \beta^* \|$. Hence $\| \beta \| \subseteq \| \beta^* \|$ is length-preserving so that

$$\lg(u) + \lg(y) = \lg(v) + \lg(z)$$

i.e., $\| \alpha\gamma \|$ is length-preserving.

We shall need the following lemma.

LEMMA 5.4. *Let a transduction $R \subseteq \Sigma^* \times \Delta^*$ be given by a transduction expression α (i.e., $R = \| \alpha \|$). There is an effective procedure for deciding whether R is length-preserving.*

Proof: Let α denote R. We first consider the case where no $*$'s occur in α. α is modified in the following systematic way.

(a) If $\beta\emptyset$ or $\emptyset\beta$ occurs as part of α where β is a transduction expression, replace $\beta\emptyset$ or $\emptyset\beta$ by $\emptyset$. Continue until there are no occurrences of $\beta\emptyset$ or $\emptyset\beta$. If $\beta \cup \emptyset$ or $\emptyset \cup \beta$ occurs in α, replace it by β. Then α is length-preserving iff the resulting expression is.

(b) Replace each (a, b) in the resulting expression by $(\lg(a), \lg(b))$.

(c) Reduce the resulting expression by the rules

$$\beta\gamma = \{(\lg(b_1) + \lg(c_1), \lg(b_2) + \lg(c_2)) \mid (\lg(b_1), \lg(b_2)) \in \beta, (\lg(c_1), \lg(c_2)) \in \gamma\}.$$

(d) Merge the resulting expression by the rule

$$\beta \cup \gamma = \{(a, b) \mid (a, b) \in \beta \quad \text{or} \quad (a, b) \in \gamma\}.$$

(e) R is length-preserving iff the set of numbers from α, call it $\hat{\alpha}$, is a subset of the equality relation. Since $\hat{\alpha}$ is finite, this is decidable.

In the general case, where α has $*$'s, proceed as follows.

1. Do step (a) above.

2. Pass to the innermost starred term of α, call it α_1^*. If there are several, work from left to right (e.g., in $(\beta_1^*\beta_2^*\beta_3^*)^*$ do β_1^* first).

3. Test α_1 for being length-preserving by steps (b)–(e). (This is valid since there are no $*$'s in α_1.)

4. If $\| \alpha_1 \|$ is length-preserving, replace α_1^* by (Λ, Λ) in α and go to step 5. If it is not length-preserving, then α is not and we are done.

5. If no $*$'s remain, use (b)–(e) again.

6. If $*$'s remain go to step 2 and continue.

Clearly, the process will terminate since the number of stars is reduced in each iteration. Using the previous lemma, the verification of the algorithm is immediate.

Example.

$$\alpha = (0, 1)^*((\Lambda, 1) \cup \emptyset)^* \cup (001, 110)$$

$$\alpha_1 = (0, 1)^*(\Lambda, 1)^* \cup (001, 110)$$

$$\alpha_2 = (\Lambda, \Lambda)(\Lambda, 1)^* \cup (001, 110).$$

Hence α is not length-preserving since $(\Lambda, 1)$ is not.

THEOREM 5.3. *Let $R \subseteq \Sigma^* \times \Delta^*$ be denoted by a transduction expression α(i.e., $R = \| \alpha \|$). It is decidable whether R is a $SR/(\Sigma \times \Delta)$.*

Proof: By Lemma 5.1, it is decidable if R is length-preserving. If R is not length-preserving, then R is not a $SR/(\Sigma \times \Delta)$. If R is length-preserving, then, by Corollary 6.6 of Elgot and Mezei (1965) R is $(\Sigma \times \Delta)$-regular. If R is $(\Sigma \times \Delta)$-regular, it is decidable whether R is a $SR/(\Sigma \times \Delta)$ by Theorem 5.1.

By the closure properties of transductions, we have the following result.

COROLLARY. *Let R, S be relations given by transduction expressions. There is an effective procedure for deciding whether $R \cup S$, $R \circ S$, R^*, R^T, and R^c are $SR/(\Sigma \times \Delta)$.*

In preparation for some undecidable questions, we recall the following result of Post (1946).

THEOREM 5.4. (Post, 1946) *Let $|\Sigma| \geqq 2$ and $(x_1, \cdots, x_n)$ and $(y_1, \cdots, y_n)$ two sequences of elements of $\Sigma\Sigma^*$. It is recursively unsolvable to decide if there is some m and some integers $i_1, \cdots, i_m$, $1 \leqq i_j \leqq n$ for $1 \leqq j \leqq m$ such that $x_{i_1} \cdots x_{i_m} = y_{i_1} \cdots y_{i_m}$.*

The following result is virtually immediate.

THEOREM 5.5. *Let $|\Sigma| \geqq 2$ and R be a $SR/(\Sigma \times \Sigma)$. Let S be a transduction (over $\Sigma^* \times \Sigma^*$). It is recursively unsolvable to determine if $R \cap S$ is empty, finite, or infinite.*

Proof: Let R be the equality relation on $\Sigma\Sigma^*$. R is clearly a $SR/(\Sigma \times \Sigma)$. Let $(x_1, \cdots, x_n)$ and $(y_1, \cdots, y_n)$ be sequences of elements of $\Sigma\Sigma^*$. Define $S' = (\bigcup_{i=1}^n \bigcup_{j=1}^n (x_i, y_j))^*$. Clearly S is a transduction. (Moreover, S is a *S-transduction* in the sense of Elgot and Mezei (1965).)

Now $S \cap R = \emptyset$ if and only if there is no solution to the Post correspondence problem. Thus it is undecidable to determine if $R \cap S = \emptyset$. Since if there is one solution to the problem stated in Theorem 5.4, there are infinitely many solutions, it is undecidable to determine if $R \cap S$ is infinite. Since $R \cap S$ is finite if and only if it is empty, it is recursively unsolvable to determine if $R \cap S$ is finite.

Remark. The proof of the previous theorem indicates that the result holds for S-transductions which are a "relatively weak" type of transduction.

We have made the point that decidability questions involving $SR/(\Sigma \times \Delta)$ depend very much on the form in which the relation is given. To illustrate this point, we now assume that the sequential relation is given by a context free grammar. (We suppose that the reader is familiar with Bar-Hillel (1964).)

THEOREM 5.6. *Let* $|\Sigma| \geqq 2$ *and let* $R \subseteq (\Sigma \times \Delta)^*$ *be given by a context free grammar* G*(i.e.,* $R = L(G)$*). It is recursively unsolvable to decide if* R *is a* $SR/(\Sigma \times \Delta)$.

Proof: Consider the case $|\Delta| = 1$. Suppose $R \subseteq (\Sigma \times \Delta)^*$ is generated by a context-free grammar. Then dom (R) is a context-free language.[16]

By Lemma 3.1, R is a $SR/(\Sigma \times \Delta)$ if and only if dom $(R) = \Sigma^*$. By Theorem 6.3(c) of Bar-Hillel (1964), it is recursively unsolvable to decide if a context-free language is equal to Σ^*. The lemma is established when $|\Delta| = 1$. If there were a decision procedure when $|\Delta| > 1$, then we could regard each $R \subseteq (\Sigma \times \{\delta\})^*$ as a relation $R \subseteq (\Sigma \times \Delta)^*$ (where $\delta \in \Delta$ and $|\Delta| > 1$) and decide if R is a $SR/(\Sigma \times \Delta)$ or not. This would contradict the case $|\Delta| = 1$.

VI. CONCLUSIONS AND OPEN PROBLEMS

During the last decade sequential functions have been extensively studied. The results of these investigations have shown them to be quite useful and well-behaved objects. We have therefore considered a generalization of sequential functions to sequential relations, relations given as the union of all sequential functions associated with a fixed sequential machine. Our results show that sequential relations have much the same properties as sequential functions.

Part of the motivation for this study has come from attempts to construct a unified theory of systems. In such a theory, some writers would make the input-output relation the fundamental object and attempt to derive the state structure from this relation.

Our results have implications in this direction. The kind of theory which can be obtained depends very much on the form in which the relation is given.

[16] Define a homomorphism φ by $\varphi(\sigma, \delta) = \sigma$ for all $(\sigma, \delta) \in \Sigma \quad \Delta$. $\varphi(L(G)) = \varphi(R) =$ dom (R) and dom (R) is context-free by the substitution theorem for context-free languages (Bar-Hillel, 1964, Theorem 3.3). A grammar generating dom (R) may be effectively constructed from G and φ.

One of the results one would require from an acceptable theory of systems is the following: "Given an input-output relation, R, it is decidable if $R = R(M)$ for some *finite* sequential machine M."

In our theory, if R is given by a transduction expression, then we test R for being length-preserving. If R passes this test, we can decide if R is a $SR(\Sigma \times \Delta)$ and construct a *finite* sequential machine M such that $R = R(M)$.

To illustrate some of the pathology of the general problem we present an example due to our colleague, Professor P. Varaiya. We wish to thank Professor Varaiya for allowing us to use his construction.

DEFINITION. Let $M_2 = \langle\{0, 1\}, \{0, 1\}, \{s_1, s_2\}, f_2, g_2\rangle$ where $f_2(s_1, 0) = f_2(s_1, 1) = s_1$, $f_2(s_2, 0) = f_2(s_2, 1) = s_2$, $g_2(s_1, 0) = 0$, $g_2(s_1, 1) = 1$, and $g_2(s_2, 0) = g_2(s_2, 1) = 0$. For each $n > 2$, define $M_n = \langle\{0, 1\}, \{0, 1\}, \{s_1, \cdots, s_n\}, f_n, s_n\rangle$ where $f_n = f_{n-1} \cup f_n'$ and $g_n = g_{n-1} \cup g_n'$. We define $f_n'(s_n, 0) = s_{n-1}$, $f_n'(s_n, 1) = s_1$, $g_n'(s_n, 0) = 0$, and $g_n'(s_n, 1) = 1$.

Since the definition of M_n is recursive, the infinite state machine $M_\omega = \langle\{0, 1\}, \{0, 1\}, S_\omega, f_\omega, g_\omega\rangle$, where $S_\omega = \bigcup_{i<\omega} s_i$, $f_\omega = \bigcup_{i<\omega} f_i$, and $g_\omega = \bigcup_{i<\omega} g_i$, is well defined. See Fig. 3.

For each $\omega \geqq n \geqq 2$, M_n has the following properties:

(i) M_n is minimal.

(ii) $R(M_n) = \{(0, 0), (1, 0)\}^* \cup \{(0, 1), (1, 1)\}^*$.

This is an example of (i) an infinite sequence of nonequivalent finite sequential machines with the same relation and (ii) an infinite minimal machine with the same relation as a minimal two state machine.

Although it is known that there is no unique minimal machine having a given relation R, it seems important to give a procedure for determining a (relationally) minimal machine from a given machine.

APPENDIX

A CONVERSATIONAL SYNTHESIS METHOD

In a lecture (July, 1964), M. A. Aizerman discussed a method for the synthesis of finite automata, but all technical details were omitted in the lecture. The method presented below represents an attempt to formulate a synthesis method which satisfied the conditions discussed by Aizerman. Recently, the method discussed by Aizerman has been published and translated into English. (Cf. (Tal, 1964).) Interestingly enough, the two methods have almost nothing in common.

At the intuitive level, the method proceeds as follows. A customer C

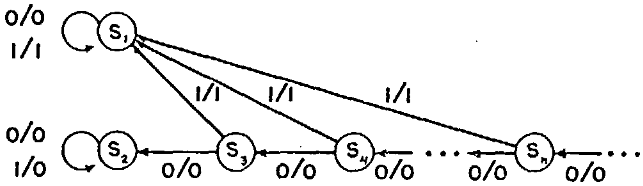


FIG. 3. A minimal infinite sequential machine which has the same sequential relation as a two state sequential machine.

brings a digital synthesis problem to a computer design expert E. The synthesis problem is to realize a given function f where f is any map[17] from $\{0, 1\}^*$ into $\{0, 1\}$. The expert asks C how much he is willing to spend to realize the function f, because if C has unbounded resources, E guarantees a solution to the problem. For the sake of realism, C is assumed to have a finite limit on the cost of the final circuit. We assume that the cost is represented by the number of internal states, which is denoted by n.

E then asks C, what is $f(0)$ and what is $f(1)$? C responds. E then demands to know $f(00), f(01), f(10), f(11)$. After C answers, E begins to ask questions of a different form. E asks, does $f(00x) = f(0x)$ for all sequences x of length at most $n - 1$? After C answers, E continues to ask a large number of questions of the same kind. After a finite length of time, E will present C with a *minimal* finite-state machine which computes f or E will announce that no such machine exists with n or fewer internal states. If no solution is obtained, then C must either give up or increase n.

We now make the intuitive discussion precise.

DEFINITION. λ_t is defined as the mapping from Σ^* to Σ^* which is given by $\lambda_t(x) = tx$ for all $x \in \Sigma^*$.

DEFINITION. A *sequential machine with initial state N* is defined as $N = \langle \Sigma, \Delta, S, M, a, N \rangle$ where $\Sigma(\Delta)$ is the *input* (*output*) alphabet, S is the set of *internal states*, $a \in S$ is the *initial state*, $M: S \times \Sigma \rightarrow S$ is the *transition function* while $N: S \times \Sigma \rightarrow \Delta$ is the *output function*.

The whole method depends on the following well-known result (Raney, 1958; Krohn-Rhodes, 1962).

THEOREM A.1. *Let f be any function from Σ^* into Δ. f is computed by*

[17] The assumption that the alphabets are binary is inessential.

the machine $N(f) = \langle \Sigma, \Delta, \{f\lambda_t \mid t \in \Sigma^*\}, M, f, N\rangle$ *where*

$$M(f\lambda_t, \sigma) = f\lambda_{t\sigma}$$

$$N(f\lambda_t, \sigma) = f(t\sigma).$$

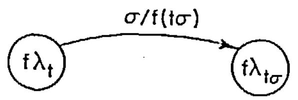
Furthermore, $N(f)$ *is the unique minimal sequential machine computing* f.

One immediately notes that such a function f is realized by a finite state machine if and only if $\{f\lambda_t \mid t \in \Sigma^*\}$ is finite.

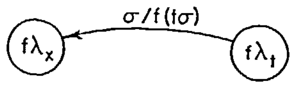
The question and answer game is formalized in the following algorithm.

*Algorithm A.*1. Given the tabulation of a function $f: \Sigma^* \to \Delta$ and an upper bound n on the number of states of a machine computing f.

1. Let f be the initial state of $N(f)$, i.e., $t = \Lambda$.
2. Compute $M(f\lambda_t, \sigma)$ for all $\sigma \in \Sigma$.
3. If $f\lambda_{t\sigma} = f\lambda_x$ for any x previously generated, go to step 5. Note that this question becomes, "is $f(t\sigma w) = f(xw)$ for all $w \in \Sigma^* - \{\Lambda\}$." One need not ask this question for every w, but only for those w whose length does not exceed $n - 1$.[18]
4. If the new state $f\lambda_{t\sigma}$ is different from all the other previous states, include it in the state diagram as shown.



5. If $f\lambda_t = f\lambda_x$, the state diagram is modified as shown.



6. Repeat the process for the words in Σ^* generated in "natural" order (e.g., $t = 0, 1, 00, 01, 10, 11, \cdots$) up to and including those of length $2n - 1$.
7. If the process terminates before n states are used, then the result is the minimal state graph.
8. If the process uses more than n states, we arbitrarily terminate the construction since either the customer has in mind a function which is

[18] This is a form of the main theorem on minimizing the number of states. Two states are equivalent if they are $(n - 1)$-equivalent in the Mealy model. (Cf. (Ginsburg, 1962).)

not realizable by a finite state machine or he is too stingy with the states.

Remark. This method requires only two types of information, the upper bound on the number of states and the first

$$\frac{|\Sigma|^{2n} - 1}{|\Sigma| - 1}$$

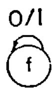
input sequences.[19]

Example of the construction: $n = 2$

t	$f(t)$	t	$f(t)$
Λ	Λ	000	1
0	1	001	0
1	0	010	0
00	1	011	1
01	0	100	0
10	0	101	1
11	1	110	1
		111	0

1. Draw 

2. $M(f, 0) = f\lambda_0$. Is $f(0x) = f(x)$ for all x of length 1? The answer is yes, $f(0) = 1$, so the diagram is

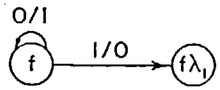


[19] If we have n states, we need all input sequences of length at most $n + (n - 1) = 2n - 1$. The number of input sequences of length at most $2n - 1$ is

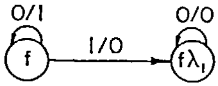
$$\begin{array}{ll} \dfrac{|\Sigma|^{2n} - 1}{|\Sigma| - 1} & \text{if } |\Sigma| > 1 \\ 2n & \text{if } |\Sigma| = 1 \end{array}$$

where $|\Sigma|$ denotes the cardinality of Σ.

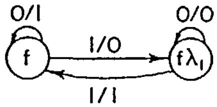
3. $M(f, 1) = f\lambda_1$, Is $f(1x) = f(x)$ for all x of length 1? The answer is no since $f(10) = 0 \neq 1 = f(0)$. Thus the state diagram becomes



4. $M(f_1, 0) = f\lambda_{10}$. Is $f(10x) = f(x)$? No using $x = 0$. Now we ask, is $f(10x) = f(1x)$ for all x of length 1? Yes. The state diagram becomes



5. $M(f_1, 1) = f\lambda_{11}$. Is $f(11x) = f(x)$? Yes. The construction is complete. The minimal machine, a mod 2 counter, is



This synthesis method is altogether different from the one presented by Tal (1964). Tal is concerned with machines that realize sequential relations and different methods are then required. (Cf. (Gill, 1966) and (Deuel and Gill, 1966).)

Acknowledgment

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